Chapter 1

Revision of basic concepts

1.1 Principles of logic

In order to establish the laws of set algebra, one can use logical deduction. Let us very briefly review the algebra developed by Boole [1] in 1854. Logical propositions must be either true or false. For convenience, the 'logical value' of any proposition can be represented by numerical values:

$$\text{“true”} = 1 \qquad \text{“false”} = 0 \qquad . \tag{1.1}$$

Given two propositions: p and q, then the *conjunction*:

$$p \wedge q$$

denotes the logical expression 'p and q'. The conjunction is *true* if, and only if, both p and q are *true*.

The *disjunction*

$$p \vee q$$

denotes 'p or q'. That is, the logical value is *true* if either p or q are *true*.

Finally the *negation* of a proposition is denoted by $\neg p$, meaning the opposite of p.

These rules of logic can be summarized in algebraic form by a *truth table*:

p	q	$p \vee q$	$p \wedge q$	$\neg p$
1	1	1	1	0
1	0	1	0	0
0	1	1	0	1
0	0	0	0	1

So, for example, the third row of the table above summarizes the following logical statements. Given that p is false ($p = 0$) and q is true ($q = 1$), in answer to the question,“ is either p or q true ?”, we say yes (true). So in the table we enter the value in the third column:

$$p \vee q = 1 \qquad .$$

While in response to the question,“ are both p and q true ?”, we say no (false). That is:

$$p \wedge q = 0 \qquad ,$$

and finally, the negation of p *is true*, given that p *is false*, and so on.

From these rules we have the *logical* expression of DeMorgan's laws [2]:

$$\neg(p \vee q) = (\neg p) \wedge (\neg q) \qquad , \qquad \neg(p \wedge q) = (\neg p) \vee (\neg q)$$

The first law can be verified by an example. Given the following popositions: p=“the ball is black” . q=“the ball is number 24”, then since $\neg(p \vee q)$=“the ball is neither black NOR is it number 24” is equivalent to the statement that “the ball is not black AND it is not numbered 24”.

[1] George Boole (1815-1864) Professor of Mathematics at University College Cork
[2] Augustus De Morgan (1806 -1871)

1.2 Events

Suppose there is a random process that can be observed/measured. In the theory of probability, the act of observation is called an *experiment*, and the result of the observation is called the *outcome*.

The *set* of all possible outcomes is called the *sample space*, or universal set. In these lectures it is denoted by Ω, but the symbol S is also commonly used.

Example A single coin is tossed and falls to the ground. The sample space of the upward face of the coin is either *heads* (H) or *tails* (T). Therefore, the sample space is:

$$\Omega = \{H, T\} \tag{1.2}$$

If the upward face of a (six-sided) die is considered the outcome, then:

$$\Omega = \{1, 2, 3, 4, 5, 6\} \tag{1.3}$$

An *event*, A, is a collection of outcomes, that is a subset of Ω:

$$A \subset \Omega \tag{1.4}$$

For the roll of a single die, one could define events such as the following:

- A_1: the die shows an even number. Thus $A_1 = \{2, 4, 6\}$.
- A_2: the die shows a prime number: $A_2 = \{2, 3, 5\}$.
- A_3: the die shows a 6: $A_3 = \{6\}$.

1.3 Unions and intersections

The 'logical' operators *and* and *or* have equivalents in set theory. The *union* of two subsets of Ω, A and B is denoted: $A \cup B$ (means all outcomes in either A or B, or both).

$$A \cup B = \{s : s \ \text{ in } \ A \ \text{ or } \ B\} \quad . \tag{1.5}$$

Note that in this expression on the right-hand-side the braces indicate the "set of elements", s (and the colon is read "such that").

The *intersection* of two subsets of Ω, A and B is denoted: $A \cap B$ (means all outcomes in both A and B), that is:

$$A \cap B = \{s : s \ \text{ in } \ A \ \text{ and } \ B\} \tag{1.6}$$

The symbol $\cup$ is the set equivalent of the logical operator $\vee$.

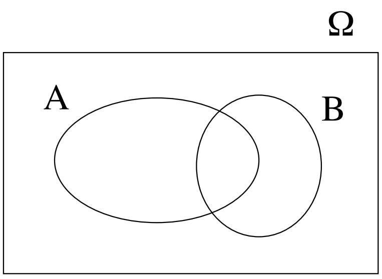

Figure 1.1: Venn diagram indicating the sets A and B

Note the following shorthand:

$$\bigcup_{i=1}^{n} A_i \equiv A_1 \cup A_2 \cup \cdots \cup A_n \quad , \quad \bigcap_{i=1}^{n} A_i \equiv A_1 \cap A_2 \cap \cdots \cap A_n \tag{1.7}$$

The *complement* of a set A, denoted by A^c:

$$A^c = \{s : s \in \Omega \text{ and } s \notin A\} \quad . \tag{1.8}$$

This has the *reflexive property*

$$(A^c)^c = A \quad . \tag{1.9}$$

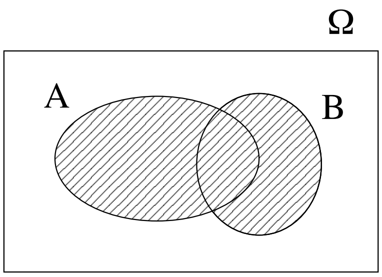

Figure 1.2: $A \cup B$: the union of A and B shaded.

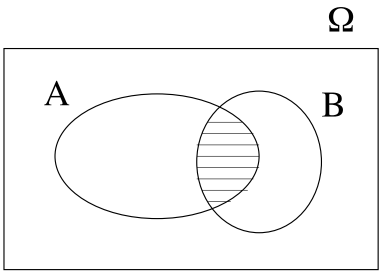

Figure 1.3: $A \cap B$: the intersection of A and B, shaded.

1.3.1 The empty set

The *empty set* is denoted by the symbol $\emptyset$ and contains no elements: it represents *impossible events*.

For example, the event:

$$B = \{s : s \text{ even} \quad \text{and} \quad s \text{ odd}\}$$

is impossible; a number cannot be both even and odd. Thus, $B = \emptyset$.
We note that the empty set plays the role of *zero* in set algebra. In particular the following very important identities, for any set A, are true:

$$A \cap \emptyset = \emptyset \quad , \quad A \cup \emptyset = A \quad . \tag{1.10}$$

By definition of an impossible event we can write:

$$A \cap A^c = \emptyset \quad \text{while} \quad A \cup A^c = \Omega \quad , \tag{1.11}$$

that is (for the first identity) an event occurring and not occurring is impossible, while an event occurring or not occurring is certain. In the language of algebra we view $\emptyset$ as the *additive identity* and thus A^c as the *additive inverse* of A.

The *difference* of two sets, is denoted by $A \setminus B$, and has the definition:

$$A \setminus B = \{s : s \in A \text{ and } \notin B\} \tag{1.12}$$

This is equivalent to the definition: $A \setminus B = A \cap B^c$.

Any two sets (A and B) such that: $A \cap B = \emptyset$ are called *disjoint sets*. The events corresponding to A and B are then said to be *mutually exclusive*; they can never both occur. That is, the occurrence of event A means that event B is excluded, and vice versa.

1.3.2 Event space

A *σ-Field* or event space, $\mathcal{F}$ is a collection of *events* (subsets) of Ω, let us denote them as $A_1, A_2, \ldots$, such that:

(a) $\emptyset \in \mathcal{F}$

(b) if $A_i \in \mathcal{F}$, then so too $A_i^c \in \mathcal{F}$

(c) if $A_1, A_2, \dots \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$

The statement (c) can be read as "$\mathcal{F}$ is closed under finite *countable unions*, and so are all intersections, unions and complements."

In the language of *group theory*, these three constraints are equivalent to the axioms of (a) the existence of an additive *identity* (b) the existence of an *inverse*, and (c) the property of *closure*. That is, we can construct rules of algebra under the operation of addition.

1.4 Set algebra

We say two sets are equal $A = B$ when:

$$A \subset B \qquad \text{and} \qquad B \subset A$$

The analogue of the *multiplicative identity* in set algebra in the sample space, Ω. Thus we have:

$$A \cap \Omega = A \qquad\qquad A \cup \Omega = \Omega$$

Having established the laws of binary operation, we can then deduce the following laws:

Commutative law:

$$A \cap B = B \cap A \qquad , \qquad A \cup B = B \cup A \tag{1.13}$$

Associative law:

$$A \cap (B \cap C) = (A \cap B) \cap C \qquad , \qquad A \cup (B \cup C) = (A \cup B) \cup C \tag{1.14}$$

We can prove these statement by appealing to the logical analogy discussed above, in which the symbols $\cap$ and $\cup$ correspond to "and" and "or". For example:

$$\omega \in A \cup (B \cup C)$$

means:

$$\omega \in A \quad \text{or} \quad \omega \in (B \cup C) \qquad ,$$

that is,

$$\omega \in A \quad \text{or} \quad \omega \in B\text{or} \quad \omega \in C \qquad .$$

Thus:

$$\omega \in (A \cup B) \quad \text{or} \quad \omega \in C$$
$$\omega \in (A \cup B) \cup C$$

Distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.15}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{1.16}$$

De Morgan Laws:

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c$$

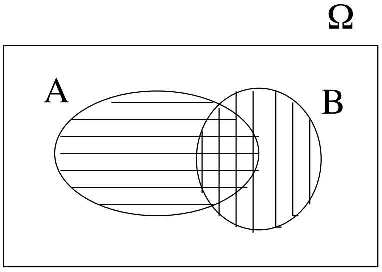

Figure 1.4: Venn diagram illustrating the inclusion-exclusion principle (equation 1.19). The union of A and B expressed as the union of three disjoint subsets. $A \setminus B$ (horizontal lines), $A \cap B$ (horizontal and vertical lines), and $B \setminus A$ (vertical lines).

with the generalization;

$$\left[\bigcup_{i=1}^{n} A_i\right]^c = \bigcap_{i=1}^{n} A_i^c \tag{1.17}$$

$$\left[\bigcap_{i=1}^{n} A_i\right]^c = \bigcup_{i=1}^{n} A_i^c \tag{1.18}$$

A very important result that describes the partitioning of the union of two sets is called the *inclusion-exclusion principle* or the *addition theorem* . It can be expressed as follows:

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B) \tag{1.19}$$

The principle defines the union of A and B in terms of the disjoint subsets: $A \setminus B$, $B \setminus A$ and $A \cap B$.

It is equivalent to the statement that, given an outcome is either in A or B, then it must be one (and only one) of the three possibilities: in A and not B, in B and not A, or in both A and B. This relation is represented graphically in figure 1.4.