Chapter 22

Inhomogeneous Poisson process

We conclude our study of Poisson processes with the case of non-stationary rates. Let us consider an arrival rate, $\lambda(t)$, that with time, but one that is still Markovian. That is we assume the system has no memory (past arrivals are not correlated with future arrivals) and disjoint time intervals are independent.

For example, a football team may score goals at a higher rate at the end of a game than at the beginning if the opposing team tires more quickly. Another example would be that text messages are sent more frequently in the evenings and afternoons than in the morning, or the arrival of customers at a shop will depend on the time of day. If we allow this flexibility in our model, and let $\lambda(t)$ vary, then we have a more powerful technique for modelling random processes. We call this the *inhomogeneous* Poisson process to distinguish it from the standard (homogeneous) process in which the rate is constant at all times.

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Let us define the inhomogeneous Poisson process as follows.

It is a counting process: $\{N(t), t \geqslant 0\}$, so that N has integer values that never decrease over time, but jump up at random times, and we specify the following 4 conditions:

(i) $N(0) = 0$

(ii) increments are independent (Markov) but *not* stationary

(iii) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$

(iv) $P(N(t+h) - N(t) > 1) = o(h)$

This is almost identical to definition 2 for the homogeneous process. Of course, we should find that any results we obtain for the inhomogeneous process will automatically reduce, if we take λ constant, to the results for the homogeneous process.

Our approach is to study how the probabilities change with time, in the manner of calculus we take an infinitesmal step forward in time and consider how the probabilities evolve during this short time in the future.

Let us first consider $P(N(t) = 0$, the probability that after a time t, *no events* have occurred, and then consider how this changes in time. We know one thing for certain: when $t = 0$, $P(N(0) = 1$, that if we start counting our event from $t = 0$, then the count is zero at that time.

$$P(N(0)) = 1 \quad , \quad \text{and} \quad P(N(t) = n) = 0 \quad , \quad n = 1, 2, 3, \ldots \quad . \tag{22.1}$$

Now consider $P(N(t) = 0)$ and $P(N(t+h) = 0)$, then we can divide the time interval $[0, t+h]$ into two disjoint time intervals $[0, t]$ and $[t, t+h]$. Then, the probability that no events occur in $[0, t+h]$ means that no events occur in $[0, t]$ and $[t, t+h]$. In mathematical notation:

$$P(N(t+h) = 0) = P(N(t) = 0 \text{ and } N(t+h) - N(t) = 0) \tag{22.2}$$

Since these are disjoint time intervals (no overlap) they are independent. Then we can write that:

$$P(N(t) = 0 \text{ and } N(t+h) - N(t) = 0) = P(N(t) = 0) \times P(N(t+h) - N(t) = 0) \quad . \tag{22.3}$$

Now using our definition above, the probability of no event happening in a very short time interval is:

$$P(N(t+h) - N(t) = 0) = 1 - \lambda(t)h + o(h) \quad . \tag{22.4}$$

So this gives us:

$$P(N(t+h) = 0) = P(N(t) = 0) \times [1 - \lambda(t)h + o(h)] \quad . \tag{22.5}$$

Let's use some shorthand, namely that $P(N(t) = 0) \equiv p_0(t)$. Then this equation can be rearranged to give:

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda(t)p_0(t) + \frac{o(h)}{h} \quad . \tag{22.6}$$

Now taking the limit $h \to 0$, gives the differential equation:

$$\frac{dp_0}{dt} = -\lambda(t)p_0 \quad . \tag{22.7}$$

This equation can be solved directly, since the variable p_0 and t are separable. We write this as:

$$\frac{dp_0}{p_0} = -\lambda(t)dt \quad . \tag{22.8}$$

and now integrate to give:

$$\int \frac{dp_0}{p_0} = -\int \lambda(t)dt \quad . \tag{22.9}$$

$$\ln p_0(t) - \ln p_0(0) = -\int_0^t \lambda(\tau)d\tau \quad . \tag{22.10}$$

But we know that $p_0(0) = 1$, so that, this gives:

$$\ln p_0(t) = -\int_0^t \lambda(\tau)d\tau \tag{22.11}$$

that is

$$\boxed{p_0(t) = e^{-\int_0^t \lambda(\tau)d\tau}} \quad . \tag{22.12}$$

Now clearly if the process is homogeneous, that is λ is a constant (does not change in time), then:

$$\int_0^t \lambda(\tau)d\tau = \lambda t$$

and this gives the familiar expression for the homogeneous case: $P(N(t) = 0) = e^{-\lambda t}$, as it should.

So consider the case $n > 0$, for $P(N(t) = n)$. This is slightly messier but repeats the same ideas. We seek $P(N(t+h) = n$ based on $P(N(t) = n)$, that is we look back in time (condition on) the earlier values.

$$P(N(t+h) = n) = \sum_{k=0}^{\infty} P(N(t+h) = n|N(t) = k) \times P(N(t) = k) \qquad . \tag{22.13}$$

Now, the conditional probabilities can be simplified:

$$P(N(t+h) = n|N(t) = k) = P(N(t+h) - N(t) = n-k) \qquad . \tag{22.14}$$

Since the count is always increasing in time we cannot have (more events in the past than in the future). This translates, mathematically, to the formula:

$$P(N(t+h) - N(t) = n-k) = 0 \qquad , \qquad k > n \qquad . \tag{22.15}$$

So the infinite sum in (22.16) is actually finite, since there are zeros for all $k > n$.

By the same token if the time step is extremely tiny $h \to 0$, there is the chance of (at most) one event happening, and most likely that no event will happen. In mathematical terms this means (using our rules)

$$\begin{aligned} P(N(t+h) - N(t) > 1) &= o(h) \\ P(N(t+h) - N(t) = 1) &= \lambda(t)h + o(h) \\ P(N(t+h) - N(t) = 0) &= 1 - \lambda(t)h + o(h) \end{aligned}$$

So only two (non-tiny) terms matter in the sum (22.16), corresponding to no extra events or one extra event. This means:

$$P(N(t+h) = n) = (1-\lambda(t)h+o(h))\times P(N(t) = n)+(-\lambda(t)h+o(h))\times P(N(t) = n-1)+o(h) \qquad . \tag{22.16}$$

Again, switching to a more convenient shorthand: $p_n(t) \equiv P(N(t) = n)$, this can be written as:

$$\frac{p_n(t+h) - p_n(t)}{h} = -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t) + \frac{o(h)}{h} \qquad . \tag{22.17}$$

Now taking the limit $h \to 0$ gives the differential equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t) \qquad , \qquad n > 0} \qquad . \tag{22.18}$$

In fact, this is a set ($n = 1, 2, 3, \dots$) of coupled first-order differential equations. In continuous time we end up with differential equations rather than diffierence equations.

22.1 Solution of the differential equations

We have the difference-differential equation:

$$\frac{d}{dt}p_n(t) = -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t) \qquad , \quad n = 1, 2, 3, \ldots \qquad . \tag{22.19}$$

and, for $n = 0$:

$$\frac{d}{dt}p_0(t) = -\lambda(t)p_0(t) \qquad . \tag{22.20}$$

The initial conditions, that define the unique solution are:

$$p_0(0) = 1 \quad , \quad p_n(0) = 0 \quad n = 1, 2, 3, \ldots \quad . \tag{22.21}$$

The equation (22.19) has two variables, the discrete (*difference*) variable n, and the continuous (*differential*) variable, t. We can rearrange (22.19), moving one term from right to left-hand side:

$$\frac{d}{dt}p_n(t) + \lambda(t)p_n(t) = \lambda(t)p_{n-1}(t) \qquad . \tag{22.22}$$

This is a first-order ordinary-differential equation in standard form, for which we have a few useful methods of solution. For example, we can use the *integrating factor* technique, that is, we multiply both sides of the equation by the term:

$$e^{\int_0^t \lambda(\tau)d\tau} \qquad , \tag{22.23}$$

giving us:

$$e^{\int_0^t \lambda(\tau)d\tau}\frac{d}{dt}p_n(t) + \lambda(t)e^{\int_0^t \lambda(\tau)d\tau}p_n(t) = \lambda(t)e^{\int_0^t \lambda(\tau)d\tau}p_{n-1}(t) \qquad . \tag{22.24}$$

This leads to the simpler form:

$$\frac{d}{dt}\left(e^{\int_0^t \lambda(\tau)d\tau}p_n(t)\right) = \lambda(t)e^{\int_0^t \lambda(\tau)d\tau}p_{n-1}(t) \tag{22.25}$$

Now, for the sake of abbreviation we define, $z_n(t) \equiv e^{\int_0^t \lambda(\tau)d\tau}p_n(t)$. Then we have the simpler difference-differential equation:

$$\boxed{\frac{d}{dt}z_n = \lambda(t)z_{n-1}(t) \quad , \quad n = 1, 2, 3, \ldots} \qquad . \tag{22.26}$$

with the equation for $n = 0$, corresponding to (22.20):

$$\frac{d}{dt}z_n = 0 \qquad . \tag{22.27}$$

This equation can be solved by *iteration*. That is, we solve (22.27) first to find $z_0(t)$, and then solve (**??**), for $z_1(t)$, using the solution from (22.27). Then solve (22.26) for $z_2(t)$, and so on an infinitum.

The solution of (22.27) with the boundary condition, $p_0(0) = 1$, is, $z_0 = 1$, that is

$$p_0(t) = e^{-\int_0^t \lambda(\tau)d\tau} \qquad , \tag{22.28}$$

as obtained before. Then, for $n = 1$,

$$\frac{d}{dt}z_1 = \lambda(t)z_0(t) = \lambda(t) \qquad , \tag{22.29}$$

This can be integrated directly, recalling the initial condition $z_1(0) = 0$. with solution:

$$z_1(t) = \int_0^t \lambda(\tau) d\tau \qquad , \tag{22.30}$$

Then, according to (22.26)

$$z_2(t) = \int_0^t \lambda(\tau) z_1(\tau) d\tau \qquad . \tag{22.31}$$

That is, substituting for $z_1(t)$, we have:

$$z_2(t) = \int_0^t \lambda(\tau) \left[\int_0^\tau \lambda(\tau') d\tau' \right] d\tau = \frac{1}{2} \left[\int_0^t \lambda(\tau) d\tau \right]^2 \qquad . \tag{22.32}$$

You can convince yourself that this is correct by calculating dz_2/dt,

$$\frac{d}{dt} \left(\frac{1}{2} \left[\int_0^t \lambda(\tau) d\tau \right]^2 \right) = \left[\int_0^t \lambda(\tau) d\tau \right] \frac{d}{dt} \left[\int_0^t \lambda(\tau) d\tau \right] = \lambda(t) \left[\int_0^t \lambda(\tau) d\tau \right] \qquad . \tag{22.33}$$

and showing that it satisfies (22.26) for $n = 2$. For $n = 3$ we have:

$$z_3(t) = \int_0^t \lambda(\tau) z_2(\tau) d\tau = \frac{1}{2 \times 3} \left[\int_0^t \lambda(\tau) d\tau \right]^3 \qquad . \tag{22.34}$$

Continuing in the same manner, expressions can be found for all n. A pattern soon emerges and we find that, for *any* value of n we have:

$$z_n(t) = \frac{1}{n!} \left[\int_0^t \lambda(\tau) d\tau \right]^n \qquad , \tag{22.35}$$

from which it follows that:

$$\boxed{p_n(t) = \frac{e^{-\int_0^t \lambda(\tau) d\tau}}{n!} \left[\int_0^t \lambda(\tau) d\tau \right]^n} \qquad . \tag{22.36}$$

Again, in case of doubt one can verify that this is the solution of (22.19) by substitution. Of course, in the special case of having a homogeneous process we get:

$$\boxed{P(N(t) = n) = \frac{e^{-\lambda t}}{n!} (\lambda t)^n} \qquad . \tag{22.37}$$

That is, we retrieve our definition 1 of a Poisson process.

Thus, according to (22.36) we have a Poisson distribution with mean (and variance) given by:

$$\boxed{\mathbb{E}\left(N(t)\right) = \int_0^t \lambda(\tau) d\tau} \tag{22.38}$$

If instead, we start counting the process at a time t_2 and stop counting at a later time $t_2 > t_1$, then the boundary conditions would mean that:

$$p_0(t_1) = 1 \quad , \quad p_n(t_1) = 0 \quad n = 1, 2, 3, \ldots \quad . \tag{22.39}$$

That is, our lower limit of integration changes but not the differential equations themselves. That is, we are simply resetting the zero of time to t_1, but this is the only change, and the analysis follows exactly the same lines. For example we get, for the probability of no events between times t_1 and t_2,

$$P(N(t_2) - N(t_1) = 0) = e^{-\int_{t_1}^{t_2} \lambda(\tau)d\tau} \quad . \tag{22.40}$$

It then follows that, the probability of exactly n events occurring between the limits is:

$$P(N(t_2) - N(t_1) = n) = \frac{e^{-\int_{t_1}^{t_2} \lambda(\tau)d\tau}}{n!} \left[\int_{t_1}^{t_2} \lambda(\tau)d\tau\right]^n \quad . \tag{22.41}$$

22.1.1 Solution by moment-generating function

One can also arrive at the solution (22.37), which we termed definition 1, through the moment-generating function. It's worth presenting this idea for didactic reasons if nothing else. Consider the counting process, $N(t)$, for the *homogeneous* Poisson process. The total time interval $[0, t]$ can be divided into m (equal) pieces and the total count will be the sum of the incremental counts over each interval. Let:

$$t_i \equiv \left(\frac{i}{m}\right) t \quad , \quad 0 \leq i \leq m-1 \quad , \tag{22.42}$$

be the start times of each interval. Then:

$$N(t) = \{N(t_1) - N(t_0)\} + \{N(t_2) - N(t_1)\} + \cdots + \{N(t) - N(t_{m-1})\} \quad . \tag{22.43}$$

These sub-counts (increments) will be independent of course, and since the intervals are very small we can use definition 2 to state that, for any integer $1 \leq i \leq m$:

$$P(N[t_{i+1}] - N[t_i] = 1) = \lambda(t/m) + o(t/m) \quad , \tag{22.44}$$

and

$$P(N[t_{i+1}] - N[t_i] = 0) = 1 - \lambda(t/m) + o(t/m) \quad . \tag{22.45}$$

Then, the moment-generating function takes the form:

$$M_{N(t)}(u) = \mathbb{E}\left(e^{uN(t)}\right) \tag{22.46}$$

Then:

$$M_{N(t)}(u) = \mathbb{E}\left(\exp u\left[\cdots + N(t_{i+1}) - N(t_i) + \cdots\right]\right) \tag{22.47}$$

There are m terms (of independent variables) in the sum in the exponential. This breaks into the product of m expectations. The general term has the form:

$$\mathbb{E}\left(\exp[u\left(N(t_{i+1}) - N(t_i)\right)]\right) = 1 - \lambda.(t/m) + \lambda.(t/m)e^u + o(t/m) \tag{22.48}$$

where we have used (22.44) and (22.45) to evaluate the expectation. This simplifies to:

$$\mathbb{E}\left(\exp[u\left(N(t_{i+1}) - N(t_i)\right)]\right) = 1 + \lambda(t/m)(e^u - 1) + o(t/m) \tag{22.49}$$

So, with m terms of this form, multiplied together, and taking the limit $m \to \infty$:

$$M_{N(t)}(u) = \lim_{m\to\infty} \left(1 + \lambda(t/m)(e^u - 1)\right)^m \tag{22.50}$$

This gives:

$$M_{N(t)}(u) = \exp\left(\lambda t(e^u - 1)\right) \tag{22.51}$$

This we recognise as the moment generation function of a Poisson variable and hence, because of the uniqueness of the moment generating function, we can immediately deduce the probability mass function for $N(t)$:

$$P(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \quad , \quad n = 0, 1, 2 \ldots \qquad , \tag{22.52}$$

which is in agreement with (22.37).

22.2 Markovian

So the solution for the inhomogenous case is:

$$p_n(t) = \mathrm{e}^{-\int_0^t \lambda(t)\mathrm{d}t}\frac{\left(\int_0^t \lambda(\tau)\mathrm{d}\tau\right)^n}{n!} \qquad . \tag{22.53}$$

Since the process is Markovian we can start the clock at any time, and reset the counter at any time. So in general:

$$\boxed{P(N(t_2) = n|N(t_1) = 0) = \mathrm{e}^{-\int_{t_1}^{t_2} \lambda(\tau)\mathrm{d}\tau} \times \frac{\left(\int_{t_1}^{t_2} \lambda(\tau)\mathrm{d}\tau\right)^n}{n!}} \qquad . \tag{22.54}$$

Of course, taking the special case of the homogenous process, λ constant, and starting from $t_1 = 0$

$$\int_0^t \lambda(\tau)\mathrm{d}\tau = \lambda t$$

and we regain the standard formula.

We can easily *verify* that (22.53) is a solution of (22.19). We note that, using the product rule:

$$\frac{d}{dt}p_n(t) = \frac{d}{dt}\left[\mathrm{e}^{-\int_0^t \lambda(\tau)\mathrm{d}\tau}\right]\frac{\left(\int_0^t \lambda(\tau)\mathrm{d}\tau\right)^n}{n!} + \mathrm{e}^{-\int_0^t \lambda(\tau)\mathrm{d}\tau}\frac{d}{dt}\left[\frac{\left(\int_0^t \lambda(\tau)\mathrm{d}\tau\right)^n}{n!}\right] \qquad . \tag{22.55}$$

Then using the result:

$$\frac{d}{dt}\int_0^t \lambda(\tau)\mathrm{d}\tau = \lambda(t)$$

we get:

$$\frac{d}{dt}\left[\mathrm{e}^{-\int_0^t \lambda(t)\mathrm{d}t}\right] = -\lambda(t)\mathrm{e}^{-\int_0^t \lambda(t)\mathrm{d}\tau} \tag{22.56}$$

and

$$\frac{d}{dt}\left[\int_0^t \lambda(\tau)\mathrm{d}\tau\right]^n = n\left[\int_0^t \lambda(\tau)\mathrm{d}\tau\right]^{n-1}\lambda(t) \qquad . \tag{22.57}$$

which gives:

$$\frac{d}{dt}p_n(t) = -\lambda(t)\left[\mathrm{e}^{-\int_0^t \lambda(\tau)\mathrm{d}\tau}\right]\frac{\left(\int_0^t \lambda(\tau)\mathrm{d}\tau\right)^n}{n!} + \lambda(t)\mathrm{e}^{-\int_0^t \lambda(\tau)\mathrm{d}\tau}\frac{\left(\int_0^t \lambda(\tau)\mathrm{d}\tau\right)^{n-1}}{(n-1)!} \qquad . \tag{22.58}$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = -\lambda(t)p_n(t) + \lambda(t)p_{n-1}(t) \qquad . \tag{22.59}$$

which is what we want.

QUESTION

The rate at which a newly-qualifed driver has accidents is a Poisson process with rate:

$$\lambda(t) = 0.1 - 0.02t \quad \text{per} \quad \text{year}$$

over the first three years, $0 \leq t \leq 3$.

Calculate the probability that the driver has no accidents in his second year ($1 \leq t \leq 2$), and the probability that he has exactly one accident in the second year.

ANSWER

We use the formula:

$$P(N(t_2) = n|N(t_1) = 0) = \mathrm{e}^{-\int_{t_1}^{t_2} \lambda(\tau)\mathrm{d}\tau} \times \frac{\left(\int_{t_1}^{t_2} \lambda(\tau)\mathrm{d}\tau\right)^n}{n!} \tag{22.60}$$

where $t_1 = 1$ and $t_2 = 2$. For no accidents $n = 0$ and for one accident $n = 1$. In either case, we need the integral:

$$\int_{t_1}^{t_2} \lambda(\tau)\mathrm{d}\tau = \int_{2}^{1} (0.10 - 0.02\tau)\mathrm{d}\tau \qquad . \tag{22.61}$$

This gives,

$$\left[0.10\tau - 0.01\tau^2\right]_1^2 = 0.16 - 0.09 = 0.07 \qquad . \tag{22.62}$$

So

$$P(N(2) = 0|N(1) = 0) = \mathrm{e}^{-0.07} \approx 0.932 \qquad . \tag{22.63}$$

Similarly, the probability of exactly one accident ($n = 1$) is:

$$P(N(2) = 1|N(1) = 0) = \mathrm{e}^{-0.07} \times 0.07 \approx 0.065 \qquad . \tag{22.64}$$

22.3 Time to arrival

Consider an inhomogeneous Poisson process, and the associated waiting times. Let T_1 denote the first arrival, then the waiting time is just the exponential process, discussed before:

$$\boxed{F_{T_1}(t) = P(T_1 \leq t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}} \qquad . \tag{22.65}$$

The corresponding probability density, $f_{T_1}(t)$, is the derivative of this function:

$$\boxed{f_{T_1}(t) = \frac{d}{dt}F_{T_1}(t) = \lambda(t)e^{-\int_0^t \lambda(\tau)d\tau}} \qquad . \tag{22.66}$$

Then the expected time for the first arrival was shown to be:

$$\mathbb{E}\left(T\right) = \int_0^{+\infty} e^{-\int_0^t \lambda(\tau)d\tau}dt \tag{22.67}$$