Chapter 21

Compound Poisson processes

Suppose we have a Poisson process, but the *value* of each event is itself random. That is we have a doubly random process, the timing is unpredictable and the value of each event is unpredictable. So for example, the visits of customers to a web-site might be Poissonian (the rate of visits to the web-site), but the amount each customer spends will also be unpredictable - so the value of the event itself is random.

Consider the thinning process discussed previously (see figure 21.1 below). We have a Poisson process of intensity λ which we will take to be telephone orders at our pizza delivery service. Suppose that we have two types of pizza: Type 1 (pepperoni) and Type 2 (margherita) and that 30% of customers order type 1, while 70% are orders for type 2. So any particular order has a probability $p = 0.3$ or being type I and $q = 0.7$ of being type II.

The counting variable for pizzas is then:

$$N(t) = N_1(t) + N_2(t) \tag{21.1}$$

Since $N(t)$ is a Poisson process with rate λ then we have:

$$P\left[N(t) = n\right] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n = 0, 1, 2 \ldots \qquad . \tag{21.2}$$

Then, since this is a Poisson distribution, we can immediately write down the *expected number* of orders and the variance in this number over a certain time t:

$$\mathbb{E}\left(N(t)\right) = \lambda t \qquad , \qquad \mathrm{var}\left(N(t)\right) = \lambda t \quad . \tag{21.3}$$

And since, both $N_1(t)$ and $N_2(t)$ are Poisson, we have that:

$$\mathbb{E}\left(N_1(t)\right) = p\lambda t \qquad , \qquad \mathrm{var}\left(N_1(t)\right) = p\lambda t \quad . \tag{21.4}$$

and

$$\mathbb{E}\left(N_2(t)\right) = q\lambda t \qquad , \qquad \mathrm{var}\left(N_2(t)\right) = q\lambda t \quad . \tag{21.5}$$

So, we can calculate on average how many pizzas are ordered over any given time. Suppose the the price of pizza 1 was $c_1 = £6$ while the price of pizza 2 was $c_2 = £4$, then (on average) how much money (payments) would our restaurant take in over a time t?

Let $Q(t)$ be the quantity of money paid during a time t. This will be stochastic variable of course. We can see this directly since:

$$Q(t) = c_1 N_1(t) + c_2 N_2(t) \quad , \tag{21.6}$$

and both $N_{1,2}(t)$ are stochastic.

It is easy to now estimate the average 'takings':

$$\mathbb{E}\left(Q(t)\right) = c_1 \mathbb{E}\left(N_1(t)\right) + c_2 \mathbb{E}\left(N_2(t)\right) = c_1 p \lambda t + c_2 q \lambda t \tag{21.7}$$

Therefore:

$$\mathbb{E}\left(Q(t)\right) = (c_1 p + c_2 q)\lambda t = \mathbb{E}\left(C\right)(\lambda t) \tag{21.8}$$

or simply:

$$\boxed{\mathbb{E}\left(Q(t)\right) = \mathbb{E}\left(C\right)\mathbb{E}\left(N(t)\right)} \quad . \tag{21.9}$$

The expected income is the expected income per pizza times the expected number of pizzas. A result that we could have anticipated.

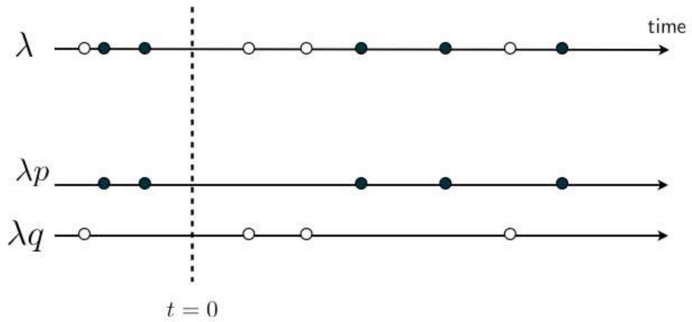


Figure 21.1: A simple compound Poisson process illustrated by a thinning process. The top line represents the random orders of pizza with a (total) rate λ. Type 1 pizzas have a probability $0 \le p \le 1$ of being ordered, and type 2 a probability $q = 1 - p$. Then each process, pepperoni pizza orders and margharita orders are themselves Poisson processes. Type I is a Poisson process with rate $\lambda_1 = \lambda p$ and Type II is Poisson with rate $\lambda_2 = \lambda(1 - p)$.

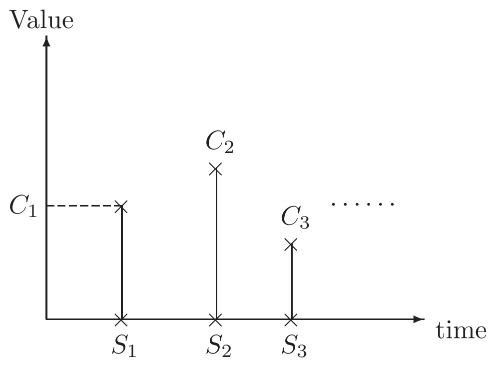
21.1 Expected value and variance

Consider a more general version of the pizza thinning process. Suppose the events we have in mind are costs/values which can have any number of values. Let us denote the cost/value of the ith event by C_i, and the number of event by time t, by $N(t)$ as usual. In this case C_i is a random variable.

The following examples illustrate what we have in mind.

	event: $N(t)$	cost: C_i
Examples:	passage of cars on a road	number of passengers per car
	arrival of customers at web-site	value of order
	insurance claims	value of claim
	infection contact	infection transmission
	cardiac arrest	degree of severity
	bankruptcy	value of loan
	score in a match	no. of points for each score

Then one could ask the same question, what would be the total expected value over a period of time $[0, t]$?



Now a major assumption: let C_i be independent and identically-distributed. Then the 'cost' of each event will have the same mean and variance:

$$\mathbb{E}(C_i) = \mu \quad , \quad \text{var}(C_i) = \sigma^2 \quad . \tag{21.10}$$

Given that the event arrivals are Poissonian, then we can make assertions about the expected number of events, and the variance in that number, by a time, t.

$$\mathbb{E}(N(t)) = \lambda t \qquad \text{var}(N(t)) = \lambda t \tag{21.11}$$

However, given that the number of events that arrive before t, which we denote as $N(t)$, is uncertain, the total value of the process:

$$Y(t) = C_1 + C_2 + C_3 + \cdots + C_{N(t)} \tag{21.12}$$

will be uncertain. Nonetheless, we can calculate the expected value of $Y(t)$ and its variance as follows.

Theorem: If $Y(t)$ is the total value of the compound Poisson process, with rate λ, and with each event having mean value $\mathbb{E}(C) = \mu$, and variance, $\text{var}(C) = \sigma^2$, then,

$$\boxed{\mathbb{E}(Y(t)) = \mu\lambda t = \mathbb{E}(C)\,\mathbb{E}(N(t))} \quad . \tag{21.13}$$

$$\boxed{\text{var}(Y(t)) = (\mu^2 + \sigma^2)\lambda t = \mathbb{E}\left(C^2\right)\text{var}(N(t))} \quad . \tag{21.14}$$

The first statement is a generalisation of the result already obtained (21.8).

Proof:

Let us first, prove the statement

$$\mathbb{E}(Y(t)) = \mu\lambda t$$

For this purpose we use the conditional expectation theorem, in which we *condition on* the number of events. That is:

$$\mathbb{E}\left(Y(t)\right) = \mathbb{E}\left(\mathbb{E}\left(Y(t)|N(t)\right)\right) \qquad . \tag{21.15}$$

where the probability mass of $N(t)$ is well-known. This becomes

$$\mathbb{E}(Y(t)) = \sum_{n=0}^{\infty} \mathbb{E}(Y(t)|N(t) = n)P(N(t) = n) \qquad . \tag{21.16}$$

This trick now solves one problem, in the term $\mathbb{E}(Y(t)|N(t) = n)$ the number of event is no longer uncertain it is *given* (or fixed) at n. In that case, we can calculate this term without difficulty:

$$\mathbb{E}\left(Y(t)|N(t) = n\right) = \mathbb{E}(C_1 + C_2 + \ldots + C_n) \tag{21.17}$$

that is,

$$\mathbb{E}\left(Y(t)|N(t) = n\right) = \mathbb{E}(C_1) + \mathbb{E}\left(C_2\right) + \cdots + \mathbb{E}\left(C_n\right) = n\mathbb{E}\left(C\right) = n\mu \qquad . \tag{21.18}$$

since the C_i all have the same mean, μ. Then it follows that:

$$\mathbb{E}\left(Y(t)\right) = \sum_{n=0}^{\infty} n\mu P(N(t) = n) = \mu \sum_{n=0}^{\infty} nP(N(t) = n) \qquad . \tag{21.19}$$

But we recognise that:

$$\sum_{n=0}^{\infty} nP(N(t) = n) = \mathbb{E}\left(N(t)\right) = \lambda t \tag{21.20}$$

for a Poisson process.

This leads to the final result:

$$\mathbb{E}\left(Y(t)\right) = \mu\lambda t \qquad . \tag{21.21}$$

Now for the variance:

$$\text{var}\left(Y(t)\right) \equiv \mathbb{E}\left(Y^2(t)\right) - \left(\mathbb{E}\left(Y(t)\right)\right)^2 \qquad . \tag{21.22}$$

In order to calculate the expectation of $Y^2(t)$, we use the same technique, conditioning on the number of events. So;

$$\mathbb{E}(Y^2) = \sum_{n=0}^{\infty} \mathbb{E}(Y^2|N(t) = n)P(N(t) = n) \qquad . \tag{21.23}$$

But

$$\mathbb{E}(Y^2|N(t) = n)) = \mathbb{E}(C_1 + C_2 + \ldots + C_n)^2 \qquad . \tag{21.24}$$

Consider the n^2 terms that arise from the expression:

$$(C_1 + C_2 + \ldots + C_n)^2 \qquad .$$

of these n^2 terms, n terms will be like: $C_1C_1, C_2C_2, \ldots, C_nC_n$, and the remaining $(n^2 - n)$ terms will look like $C_iC_j \qquad (i \neq j)$.

Now the C_i are identically distributed so:

$$\mathbb{E}\left(C_1^2\right) = \mathbb{E}\left(C_2^2\right) = \cdots = \mathbb{E}\left(C_n^2\right) = \sigma^2 + \mu^2 \qquad . \tag{21.25}$$

This follows since: $\mathrm{var}\,(C) = \sigma^2 = \mathbb{E}\left(C^2\right) - \mu^2$.

Moreover, the C_i are independent, so that:

$$\mathbb{E}(C_i C_j) = \mathbb{E}(C_i)\mathbb{E}(C_j) = \mu^2 \qquad , \qquad i \neq j \qquad . \tag{21.26}$$

With this in mind:

$$\mathbb{E}(C_1 + C_2 + \ldots + C_n)^2 = n(\sigma^2 + \mu^2) + (n^2 - n)\mu^2 = n\sigma^2 + n^2\mu^2 \qquad . \tag{21.27}$$

So returning to the calculation:

$$\mathbb{E}(Y^2) = \sum_{n=0}^{\infty}(n\sigma^2 + n^2\mu^2)P(N(t) = n) = \sigma^2\mathbb{E}\left(N(t)\right) + \mu^2\mathbb{E}\left(N^2(t)\right) \qquad . \tag{21.28}$$

Then the variance is just:

$$\mathrm{var}\,(Y(t)) = \mathbb{E}\left(Y(t)\right) - \left(\mathbb{E}\left(Y(t)\right)\right)^2 = \sigma^2\mathbb{E}\left(N(t)\right) + \mu^2\mathbb{E}\left(N^2(t)\right) - \mu^2(\mathbb{E}\left(N(t)\right))^2 \tag{21.29}$$

and this reduces to:

$$\mathrm{var}\,(Y(t)) = \sigma^2\mathbb{E}\left(N(t)\right) + \mu^2\mathrm{var}\left(N(t)\right) \tag{21.30}$$

which in turn, simplifies to:

$$\mathrm{var}\,(Y(t)) = \sigma^2\lambda t + \mu^2\lambda t = \lambda t(\mu^2 + \sigma^2) \qquad . \tag{21.31}$$

So returning to the shop, the standard deviation on the takings can be calculated using the formula (21.14). Recall that customer visit at a rate $\lambda = 4$ per hour, with $\mu = £10.00$ and $\sigma^2 = 20.00(£^2)$. Then over a two-hour period the variance of the total takings would be:

$$\mathrm{var}(\mathrm{Takings}) = 4 \times 2 \times (10^2 + 10) = 960(£^2)$$

and then the standard deviation is just the square root of this number: £30.98

21.2 Compound Poisson processes and money

Let's conclude this chapter, and indeed this section, with a rather specialised application of Poisson processes to money. In this case, we are thinking of the financial risk to a bank, or business, over a period of many months. The purpose of this section is to, as usual, estimate expected values and standard deviations of a stochastic process. The process we have in mind is a compound (inhomogeneous) process, specifically insurance claims on an insurance company. Let us again assume that the amount/value of the ith claim is C_i, and where C_i are independent and identically distributed random variables. We assume that the claims arise/occur in the manner of an inhomogeneous Poisson process with a rate $\lambda(t)$.

21.2.1 Time-value of money

Now, in the world of money - costs and values are not always what they seem. For example £100 now, if I invest it in a bank, will be worth £110 a year from now if I am lucky enough to find a bank that pays 10% annual equivalent interest. Similarly, a bill for £50 due in 6 months time, is worth less than £50 in today's prices, for the same reason. I could cover that bill if I put aside say £48 today (in a savings account).

This idea is called the *time value* of money. We say that future costs can be discounted (devalued if you like) if we wish to calculate their present-day value, and we do this on the basis of (risk-free) interest-bearing investments. We can calculate the discounting factor by considering savings. In *compound interest*, if we put an amount of money V_0 into an investment that returns interest m times per year, then after 1 year it will be worth:

$$V_1 = V_0 \left(1 + \frac{r}{m}\right)^m \tag{21.32}$$

where r is the 'interest rate'. This r is related to the annual equivalent rate (AER) R by:

$$(1 + R)V_0 = V_0 \left(1 + \frac{r}{m}\right)^m \tag{21.33}$$

that is

$$R = \left(1 + \frac{r}{m}\right)^m - 1 \approx r \qquad , \qquad (r/m) \ll 1 \tag{21.34}$$

Then over t years we have:

$$V(t) = V_0 \left[\left(1 + \frac{r}{m}\right)^m\right]^t \tag{21.35}$$

Suppose m were large, many frequent interest payments (perhaps on a weekly/daily rate) then:

$$\left(1 + \frac{r}{m}\right)^m \approx e^r \qquad , \qquad m \gg 1 \quad . \tag{21.36}$$

so that:

$$V(t) \approx V_0 e^{rt} \tag{21.37}$$

and we call this the continuous-compounding approximation, and r the annual continuously-compounded interest rate (called *force of interest* in insurance). So this formula tells us (approximately) the value of an investment, worth V_0 now, in t years time from now. So £100 invested now with $r = 0.02$ (per annum), after 6 months ($t = 0.5$ years) will be worth £101 approximately.

We can use the same formula to work out the (future) cost of a bill in today's prices, by reversing the formula. A bill of £202 due for payment in 6 months time is worth £200 pounds at today's prices. We can see this, because if we save £200 now, in 6 months time we can cover the bill. Thus, future costs can be discounted to find their present-day value. So if the future cost (at time t) is C, the present-day cost is:

$$\boxed{C_0 = Ce^{-rt}} \quad . \tag{21.38}$$

We call e^{-rt} the *discount factor*.

21.2.2 The expected cost

Suppose the bills/costs, let's call them insurance claims but they could be mortgage loan defaults or something similar. Then one possible question is - how much money do I need to have now in order to cover these costs ? Well, since this is a random process we cannot answer that question. However, we can answer the related questions - approximately how much do I need, and what is the error in that approximation. The equivalent mathematical quantities are the *expected value* and the *standard deviation*. The expected value is the best estimate, and the standard deviation is a measure of the error in that estimate. In financial terms this error is the amount of 'risk' I'm taking. The risk of running out of money and going bust!

So, the *present-day* value of all these future claims, up until at time t, can be written as:

$$Z(t) = C_1 e^{-rt_1} + C_2 e^{-rt_2} + \cdots + C_{N-1} e^{-rt_{N-1}} + C_N e^{-rt_N} \tag{21.39}$$

where $N(t)$ is the (random) number of events/claims/costs that arise, and $t_1, t_2, \ldots$ are the (random) times at which these arise, and C_i are the costs accruing when they arise. The times, are of course unpredictable, being dictated by an inhomogeneous Poisson process. Then the number of events, $N(t)$ is also unpredictable.

Let's calculate $\mathbb{E}(Z(t))$ using the usual trick of *conditional expectation*. In fact, the problem is just the same as the frog crossing the road. We only have to think about what happens to the frog in a short interval of time.

$$\mathbb{E}(Z(t+h)) = \mathbb{E}(\mathbb{E}(Z(t+h)|X)) \tag{21.40}$$

where X is the event in the time interval $(t, t+h)$. Denoting $z(t) = \mathbb{E}(Z(t))$, and A being the event that a claim occurs and A^c being an event that a claim does not occur, in this interval we have:

$$\mathbb{E}(Z(t+h)) = \mathbb{E}(Z(t+h)|A^c)\, P(A^c) + \mathbb{E}(Z(t+h)|A)\, P(A) \tag{21.41}$$

Then this can be written as:

$$\mathbb{E}(Z(t+h)) = \mathbb{E}(Z(t))\, P(A^c) + \mathbb{E}\left(Z(t) + Ce^{-rt}\right) P(A) \qquad .(21.42)$$

that is:

$$z(t+h) = z(t)\left(1 - \lambda h + o(h)\right) + \left(z(t) + \mu e^{-rt}\right)\left(\lambda h + o(h)\right) \qquad . \tag{21.43}$$

which gives us, taking $h \to 0$:

$$\frac{dz(t)}{dt} = \lambda \mu e^{-rt} \qquad . \tag{21.44}$$

This can be integrated directly, and using the fact that $z(0) = 0$, we get:

$$z(t) = \frac{\lambda\mu}{r}\left(1 - e^{-rt}\right) \tag{21.45}$$

that is

$$\boxed{\mathbb{E}(Z(t)) = \frac{\lambda\mu}{r}\left(1 - e^{-rt}\right)} \tag{21.46}$$

and whenever $rt \to 0$, that is there is zero interest rate, or we are thinking in the short term:

$$\mathbb{E}(Z(t)) \approx \lambda\mu t \tag{21.47}$$

as before (21.13). Now consider the case $rt \gg 1$, that is interest rates are extremely large, or we are thinking about the long-term, then effectively we have:

$$\mathbb{E}(Z(t)) \approx \frac{\lambda\mu}{r} \qquad . \tag{21.48}$$

This tells us how much money we should hold now to cover us in the long term.

As for the variance in these costs, we apply the same idea yet again. Let: $W(t) = Z(t)^2$, and $w(t) = \mathbb{E}(W(t))$, then:

$$\mathbb{E}(W(t+h)) = \mathbb{E}(W(t+h)|A^c)\, P(A^c) + \mathbb{E}(W(t+h)|A)\, P(A) \quad . \tag{21.49}$$

Then after a similar calculation we get

$$\mathbb{E}(W(t+h)) = \mathbb{E}(W(t))\, P(A^c) + \mathbb{E}\left((Z(t) + Ce^{-rt})^2\right) P(A) \quad , \tag{21.50}$$

giving us:

$$w(t+h) = w(t)\,(1 - \lambda h + o(h)) + \left(w(t) + 2\mu e^{-rt} z(t) + \mathbb{E}\left(C^2\right) e^{-2rt}\right)(\lambda h + o(h)) \tag{21.51}$$

This simplifies to:

$$\frac{d}{dt} w(t) = 2\mu\lambda e^{-rt} z(t) + \mathbb{E}\left(C^2\right) \lambda e^{-2rt} \quad . \tag{21.52}$$

We note that, from (21.45):

$$\frac{dz(t)}{dt} = \mu\lambda e^{-rt} \quad , \tag{21.53}$$

so this equation can be written as:

$$\frac{d}{dt} w(t) = \frac{d}{dt} [z(t)]^2 + \mathbb{E}\left(C^2\right) \lambda e^{-2rt} \tag{21.54}$$

And this equation can be integrated immediately, bearing in mind the initial conditions, $w(0) = z(0) = 0$. This leads to the final result:

$$w(t) = z(t)^2 + \mathbb{E}\left(C^2\right) \frac{\lambda}{2r} \left[1 - e^{-2rt}\right] \tag{21.55}$$

Since, $\mathrm{var}(Z(t)) \equiv \mathbb{E}\left(Z(t)^2\right) - (\mathbb{E}(Z(t)))^2 = w(t) - z(t)^2$, then it follows that:

$$\boxed{\mathrm{var}(Z(t)) = \mathbb{E}\left(C^2\right) \frac{\lambda}{2r} \left[1 - e^{-2rt}\right]} \quad . \tag{21.56}$$

21.2.3 Example

Suppose ticket sales at an airline arrive as a Poisson process, with rate λ. The airline decide that it will increase its prices exponential as the departure date approaches. Therefore, if the current time is $t = 0$ and the date of departure is $t = T$, the cost of a ticket, bought at a time $t > 0$ will be,

$$Ce^{rt} \tag{21.57}$$

Calculate the expected revenue (sales income) from ticket sales in this case.

This problem is very similar to the insurance problem in which the future *costs* are discounted, except in this case the payments escalate. We have Ce^{rt} instead of Ce^{-rt}. That is, mathematically all we are doing is changing the sign of r. So we can use this fact to get the final answer. In this case, the ticket price C_0 is not a random variable so we can simply say that:

$$\mu = \mathbb{E}(C) = C_0$$

Let's start with the final answer (21.46) and change $r \to -r$, and given we have the time of departure T. then we have, the expected sales revenue from tickets using this pricing model:

$$\boxed{\mathbb{E}\left(Z(T)\right) = \frac{\lambda C_0}{r}\left(e^{rT} - 1\right)} \quad . \tag{21.58}$$

Now if the pricing rise is very slow: $e^{rT} \approx 1 + rT$ and thus:

$$\mathbb{E}\left(Z(T)\right) \approx C_0 \lambda T \tag{21.59}$$

But if $rT \gg 1$ then the revenue is much greater than this figure,

$$\mathbb{E}\left(Z(T)\right) \approx \frac{\lambda C_0}{r} e^{rT} \tag{21.60}$$